A Note on Modeling Mixing in Stably Stratified Flows

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ABSTRACT

In a recent paper, Canuto et al. made a crucial contribution to modeling mixing in stably stratified flows by discovering that a modification to one of the closure constants can push the critical gradient Richardson number $\mathrm{Ri_{CR}}$, beyond which turbulence is extinguished, to infinity. In this note, following their approach, the Kantha model is modified to yield a value of infinity for $\mathrm{Ri_{CR}}$. The results are in good agreement with both the Canuto et al. results and the data presented in their paper.

1. Introduction

Following recent contributions to the development of turbulence models without a critical gradient Richardson number $\mathrm{Ri_{CR}}$ (see Sukoriansky et al. 2005, Galperin et al. 2007, and Zilitinkevich et al. 2007, 2008), Canuto et al. (2008a, hereafter CCHE) have made a crucial contribution to modeling mixing in stably stratified flows in the oceans and the atmosphere by discovering that a rather simple modification to one of the closure constants can push $\mathrm{Ri_{CR}}$, beyond which turbulence is extinguished, to infinity. This advance brings second-moment closure (SMC) models into compliance with a bunch of laboratory and field observational data as well as direct numerical simulation (DNS) studies (e.g., Galperin et al. 2007; see CCHE for a list), which appear to suggest that turbulence can exist up to $\mathrm{Ri} \sim O(100)$. Prior to this advancement (e.g., Mellor and Yamada 1982; Kantha and Clayson 1994, henceforth KC; Cheng et al. 2002, henceforth C02; Kantha 2003a; Umlauf and Burchard 2003), $\mathrm{Ri_{CR}}$ value in SMC models was $O(1)$ or well below unity. In this note, we follow the lead of CCHE and modify the Kantha (2003a, henceforth K03) model to yield a value of infinity for $\mathrm{Ri_{CR}}$ and demonstrate that the results in this paper are in good agreement with CCHE's results and data. We conclude with a few cautionary comments on the topic.

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2. The basic model

For completeness, we will start with the governing equations that are a simplified form of those presented in appendix B of CCHE. The simplification results from the fact that if we neglect the Coriolis terms in the equations for second-moment quantities, without any loss of generality, we can orient the x axis in the direction of the mean flow so that the algebra involved in deriving the equations for mixing coefficients becomes much more tractable (e.g., K03). The three components of the turbulent kinetic energy (TKE) are given by (the notation is as in CCHE)

$$\overline{u^2} = \frac{q^2}{3} - \frac{\tau}{3}\left[(\lambda_2 + 3\lambda_3)\overline{uw}\frac{\partial U}{\partial z} + 2\lambda_4 g\alpha_T\overline{w\theta}\right],$$

$$\overline{v^2} = \frac{q^2}{3} - \frac{\tau}{3}\left(-2\lambda_2\overline{uw}\frac{\partial U}{\partial z} + 2\lambda_4 g\alpha_T\overline{w\theta}\right), \quad \text{and} \qquad (1)$$

$$\overline{w^2} = \frac{q^2}{3} + \frac{\tau}{3}\left[(-\lambda_2 + 3\lambda_3)\overline{uw}\frac{\partial U}{\partial z} + 4\lambda_4 g\alpha_T\overline{w\theta}\right].$$

The rest of the Reynolds stresses are

$$\overline{uv} = 0, \quad \overline{vw} = 0, \quad \text{and}$$
$$\overline{uw} = -\frac{\tau}{2}\frac{\partial U}{\partial z}\left[\frac{1}{2}\left(\lambda_1 - \frac{4}{3}\lambda_2\right)q^2 + (\lambda_2 - \lambda_3)\overline{u^2} + (\lambda_2 + \lambda_3)\overline{w^2}\right] + \lambda_4 \tau g \alpha_T \overline{u\theta}. \quad (2)$$

The horizontal turbulent heat fluxes are

$$\overline{u\theta} = -\frac{\tau}{\lambda_5}\left[\overline{uw}\frac{\partial \Theta}{\partial z} + \frac{1}{2}(\lambda_6 + \lambda_7)\overline{w\theta}\frac{\partial U}{\partial z}\right] \quad \text{and}$$
$$\overline{v\theta} = 0. \quad (3)$$

The vertical turbulent heat flux, along with the temperature variance that occurs in its equation, is given by

$$\overline{w\theta} = -\frac{\tau}{\lambda_5}\left[\overline{w^2}\frac{\partial \Theta}{\partial z} + \frac{1}{2}(\lambda_6 - \lambda_7)\overline{u\theta}\frac{\partial U}{\partial z} - \lambda_0 g \alpha_T \overline{\theta^2}\right] \quad \text{and}$$
$$\overline{\theta^2} = -\frac{\lambda_8 \tau}{\lambda_0}\overline{w\theta}\frac{\partial \Theta}{\partial z}, \quad (4)$$

where $\tau = 2K/\varepsilon = B_1 \ell / q$ is the turbulence dissipation time scale, ε is the dissipation rate of the TKE K $(= q^2/2)$, and ℓ is the turbulence macroscale. Equations (1)–(4) need to be solved with a differential or algebraic equation for TKE.

Note that the vertical momentum and heat fluxes are given by

$$-\overline{uw} = K_M \frac{\partial U}{\partial z} \quad \text{and} \quad -\overline{w\theta} = K_H \frac{\partial \Theta}{\partial z},$$

where the mixing coefficients K_M and K_H are, in terms of structure functions S_M and S_H, as follows:

$$K_M = K\tau S_M \quad \text{and} \quad K_H = K\tau S_H.$$

Here, B_1 and $\lambda_0 \cdots \lambda_8$ are closure constants, whose values were chosen by C02 and CCHE to be

$$\lambda_0 = \frac{2}{3}, \quad \lambda_1 = 0.107, \quad \lambda_2 = 0.0032, \quad \lambda_3 = 0.0864,$$
$$\lambda_4 = 0.1, \quad \lambda_5 = 11.04, \quad \lambda_6 = 0.786, \quad \lambda_7 = 0.643,$$
$$\lambda_8 = 0.547, \quad \text{and} \quad B_1 = 19.3.$$

Note that $\lambda_6 \neq \lambda_7$. This complicates the algebra and leads to the complicated expressions S_M and S_H in the CCHE model. If $\lambda_6 = \lambda_7$, then the shear terms vanish in the set of Eq. (4); therefore, these equations can be solved independent of Eqs. (1)–(3), and S_H can be expressed in terms of $\overline{w^2}/K$, which can then be obtained from Eqs. (1)–(3). This is the approach pioneered recently by Canuto et al. (2008b) in deriving an SMC model for double-diffusive mixing combined with conventional shear-driven turbulence. This is the same approach we follow here because in the KC and K03 models $\lambda_6 = \lambda_7$. The closure constants in K03 are

$$\lambda_0 = \frac{2}{3}, \quad \lambda_1 = 0.1239, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0.1050,$$
$$\lambda_5 = 8.9209, \quad \lambda_6 = \lambda_7 = 0.5709, \quad \lambda_8 = 0.5801, \quad \text{and}$$
$$B_1 = 16.6,$$

while those in KC are

$$\lambda_0 = \frac{2}{3}, \quad \lambda_1 = 0.168, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0.166, \quad \lambda_5 = 7.48,$$
$$\lambda_6 = \lambda_7 = 1.0, \quad \lambda_8 = 0.487, \quad \text{and} \quad B_1 = 16.6.$$

Another salient difference between the KC and K03 models and the CCHE model is that they use the quasi-equilibrium approach of Galperin et al. (1988), so that Eq. (1) is replaced by

$$\overline{u^2} = \frac{q^2}{3}(1 - 2\lambda_4),$$
$$\overline{v^2} = \frac{q^2}{3}(1 - 2\lambda_4), \quad \text{and}$$
$$\overline{w^2} = \frac{q^2}{3}(1 - 3\lambda_3 + \lambda_2) + \frac{\tau}{3}[(-\lambda_2 + 3\lambda_3 + 4\lambda_4)g\alpha_T \overline{w\theta}]. \quad (5)$$

Now, to the solution of Eqs. (2)–(5). From Eq. (4):

$$S_H = \frac{\overline{w^2}}{K}(\lambda_5 + \lambda_8 G_H)^{-1}. \quad (6)$$

From Eqs. (2), (3), and (5):

$$\frac{\overline{w^2}}{K} = \frac{2(1 - 3\lambda_3 + \lambda_2)(\lambda_5 + \lambda_8 G_H)}{3\lambda_5 + (-\lambda_2 + 3\lambda_3 + 4\lambda_4 + 3\lambda_8)G_H} \quad (7)$$

and

$$S_M\left[1 + \frac{\lambda_4}{\lambda_5}G_H - \frac{(\lambda_2 - \lambda_3)}{6}(\lambda_2 + 3\lambda_3 + 2\lambda_4)G_M\right] = \frac{1}{2}\left(\lambda_1 - \frac{4}{3}\lambda_2\right) + \frac{1}{3}(1 - 2\lambda_4)(\lambda_2 - \lambda_3) + \frac{(\lambda_2 + \lambda_3)}{2}\frac{\overline{w^2}}{K} - \frac{(\lambda_6 + \lambda_7)}{2}\frac{\lambda_4}{\lambda_5}S_H G_H, \quad (8)$$

with $G_H = \tau^2 N^2 = \tau^2 \alpha_T (\partial \Theta / \partial z)$ and $G_M = \tau^2 (\partial U / \partial z)^2$.

In both the KC and K03 models, $\lambda_2 = \lambda_3$, so that the third term in the square brackets on the left-hand side of Eq. (8) vanishes; therefore, G_M drops out, leading to the following equations for the structure functions S_M and S_H:

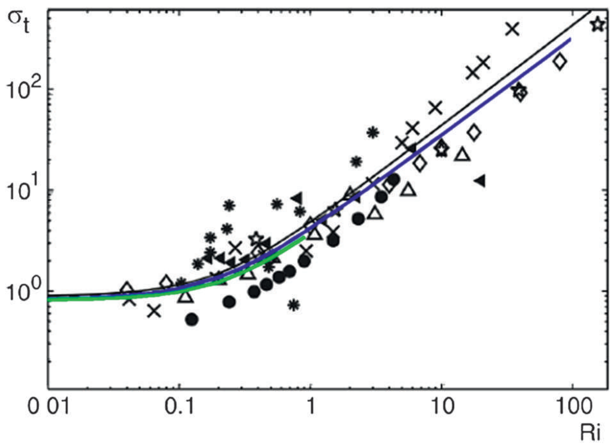

FIG. 1. The turbulent Prandtl number as a function of the gradient Richardson number Ri. The dashed and solid black lines are from the CCHE model. The green (blue) line is for K03 (K03 extended) model. The symbols denote data. See CCHE for a list of the data and their sources.

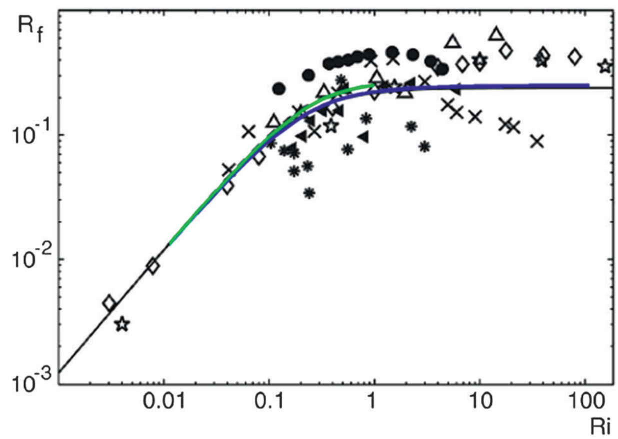

FIG. 2. As in Fig. 1, but for R_f.

$$S_M\left(1+\frac{\lambda_4}{\lambda_5}G_H\right)=\frac{1}{2}\left(\lambda_1-\frac{4}{3}\lambda_2\right)+\frac{1}{2}(\lambda_2+\lambda_3)\lambda_5 S_H + \frac{(\lambda_2+\lambda_3)\lambda_8}{2}S_H G_H - \frac{(\lambda_6+\lambda_7)}{2}\frac{\lambda_4}{\lambda_5}S_H G_H \quad \text{and} \tag{9}$$

$$S_H=\frac{2(1+\lambda_2-3\lambda_3)}{3\lambda_5+(-\lambda_2+3\lambda_3+4\lambda_4+3\lambda_8)G_H}. \tag{10}$$

Equations (9) and (10) can be used along with differential equations for K (or q^2) and a quantity containing the turbulence length scale (e.g., $q^2\ell$ or ε) in the so-called two-equation models of turbulence (e.g., KC and K03). Even when the x axis is not aligned with the mean flow, the expressions hold if $(\partial U/\partial z)^2$ is replaced by $[(\partial U/\partial z)^2 + (\partial V/\partial z)^2]$.

If the tendency, diffusion, and advection terms are ignored in the TKE equation (as in C02), it can be written as

$$S_M G_M - S_H G_H = 2. \tag{11}$$

Using Eqs. (9)–(11) results in the so-called super-equilibrium model, in which all turbulence quantities become functions of the gradient Richardson number

$$\mathrm{Ri} = N^2 S^{-2}, \tag{12}$$

because Eq. (11) can be written as $[(S_M/\mathrm{Ri}) - S_H]G_H = 2$ and used to replace G_H in Eqs. (9) and (10) by Ri. Note that N is the buoyancy frequency and S is the shear frequency equal to mean shear $(\partial U/\partial z)$.

3. The extended model and results

It is important to note that the closure constants $\lambda_0 \cdots \lambda_8$ are indeed constants. However, this leads to a finite value for $\mathrm{Ri_{CR}}$, beyond which turbulence is extinguished. This value is 0.21 for KC, 0.894 for K03, and 0.961 for the C02 models. In a flash of insight, CCHE demonstrated that by replacing the closure constant λ_5 by $\overline{\lambda_5} = \lambda_5(1+\mathrm{Ri})$, the value of $\mathrm{Ri_{CR}}$ can be extended to infinity. To do so is still within the SMC philosophy, in that the closure constants can in fact be functions of salient nondimensional parameters, such as Ri and the Reynolds number Re, if need be. We, therefore, follow the lead of CCHE and extend the K03 model. We compute various turbulence quantities as functions of Ri and, in Figs. 1–7 , compare them with CCHE results, extracted from their paper. In all the figures, the dashed and solid black lines are from the CCHE model. In Figs. 1–5, the green (blue) line is for K03 (K03 extended) model; in Figs. 6 and 7, the dash-dotted (dotted) line is for K03 (K03 extended) model.

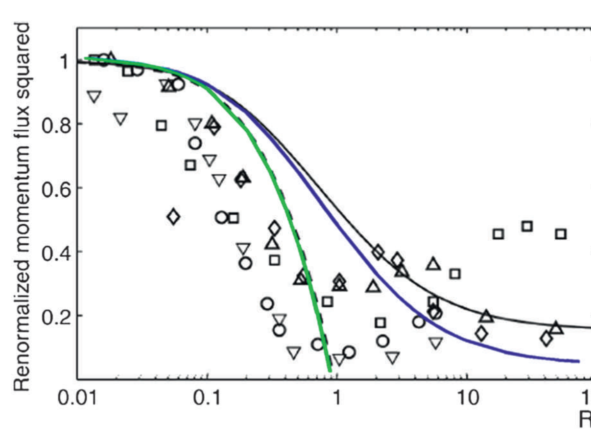

FIG. 3. As in Fig. 1, but for the square of the normalized momentum flux.

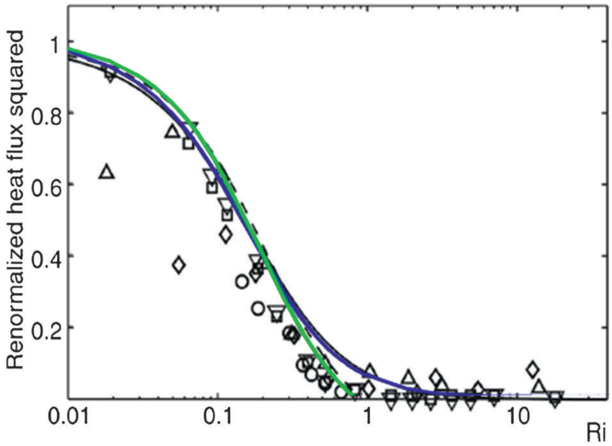

FIG. 4. As in Fig. 1, but for the square of the normalized heat flux.

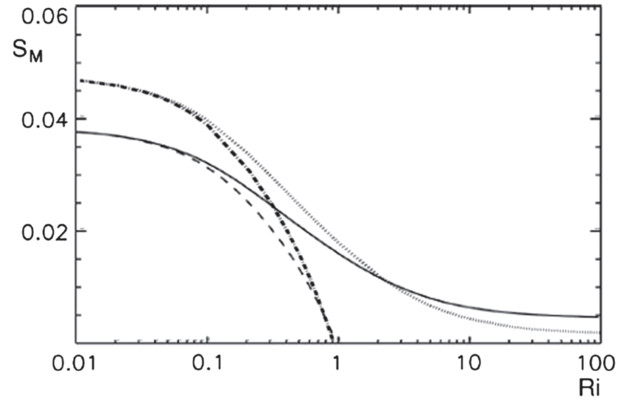

FIG. 6. The momentum flux structure function S_M as a function of the gradient Richardson number Ri. The dashed and solid black lines are from CCHE model. The dashed–dotted (dotted) line is for K03 (K03 extended) model.

Figures 1 and 2 show the turbulent Prandtl number $\sigma_t = S_M/S_H$ and the flux Richardson number $R_f = \mathrm{Ri}/\sigma_t$ as functions of Ri. The agreement of the extended K03 model with CCHE and the data cited in there (see CCHE for the data sources) is quite good. Figures 3 and 4 show the normalized squared momentum flux $(-\overline{uw}/K)^2/(-\overline{uw}/K)^2_{\mathrm{Ri}=0}$ and the heat flux $[(-\overline{w\theta})^2/(K\overline{\theta^2})]/[(-\overline{w\theta})^2/(K\overline{\theta^2})]_{\mathrm{Ri}=0}$, for which also data exist for comparison with models (note that CCHE omit mentioning that the normalized fluxes have been divided by their values at Ri= 0). Once again the agreement of the extended model with CCHE and the data cited in there is quite good. Figure 5 shows the normalized vertical velocity variance $-\overline{w^2}/(2K)$ as a function of Ri. The extended model agrees quite well with CCHE. Finally, Figs. 6 and 7 compare structure functions S_M and S_H from the extended K03 model with those from CCHE.

Note that we have not shown results from the original or extended KC models. It turns out that the replacement of λ_5 by $\overline{\lambda_5} = \lambda_5(1 + \mathrm{Ri})$ fails here to extend $\mathrm{Ri}_{\mathrm{CR}}$ from 0.21 to infinity, suggesting that perhaps the success of the technique depends on having the $\mathrm{Ri}_{\mathrm{CR}}$ of the original model be $O(1)$. This condition is satisfied by both the C02 and K03 models.

4. Concluding remarks

The objective of this note is to demonstrate that the K03 model extended along the lines suggested by CCHE pushes its $\mathrm{Ri}_{\mathrm{CR}}$ from a value $O(1)$ to infinity and provides results in good agreement with the CCHE model and the data presented therein. Although there is a philosophical difference underpinning the two modeling approaches (see Cheng et al. 2003 and Kantha 2003b), the agreement between the two models is remarkable.

It is also worth pointing out that although there is an intellectual satisfaction in bringing the SMC models to compliance with existing data, in practice, the difference between $\mathrm{Ri}_{\mathrm{CR}}$ being $O(1)$ and infinity may not have much practical impact, when these models are included

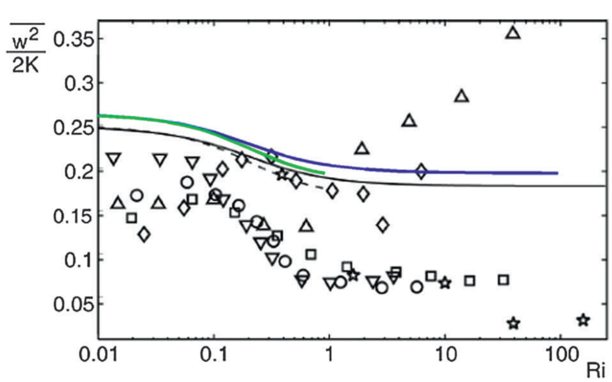

FIG. 5. As in Fig. 1, but for the ratio $\overline{w^2}/q^2$.

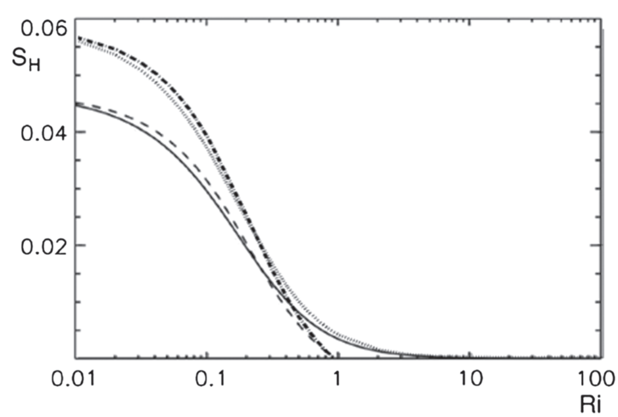

FIG. 7. As in Fig. 6, but for the heat flux structure function S_H.

in atmospheric and oceanic general circulation models (GCMs). This is because the turbulent momentum flux and heat flux become rather small beyond $\mathrm{Ri} = 1$, as can be seen from Figs. 3 and 4. This is more true for the heat flux, which nearly vanishes. There is, however, considerable residual momentum flux (but without much heat flux) beyond $\mathrm{Ri} = 1$, which suggests that the contribution to these fluxes might have come from internal waves. Given the difficulty of distinguishing between weak turbulence and internal waves, this issue is not easy to resolve. Nevertheless, all this suggests that currently popular mixed layer models with finite $\mathrm{Ri}_{\mathrm{CR}}$ can continue to be used either alone or in GCMs, without a great penalty, as long as $\mathrm{Ri}_{\mathrm{CR}}$ is $O(1)$. For a recent review of second-moment closure models, see Umlauf and Burchard (2005; see also Kantha 2006).

For prior contributions to the development of turbulence models without a critical gradient Richardson number, see Sukoriansky et al. (2005), Galperin et al. (2007), and Zilitinkevich et al. (2007, 2008).

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REFERENCES

Canuto, V. M., Y. Cheng, A. M. Howard, and I. N. Esau, 2008a: Stably stratified flows: A model with no Ri(cr). *J. Atmos. Sci.,* **65,** 2437–2447.

——, ——, and ——, 2008b: A new model for Double Diffusion + Turbulence. *Geophys. Res. Lett.,* **35,** L02613, doi:10.1029/2007GL032580.

Cheng, Y., V. M. Canuto, and A. M. Howard, 2002: An improved model for the turbulent PBL. *J. Atmos. Sci.,* **59,** 1550–1565.

——, ——, and ——, 2003: Comments on "An improved model for the turbulent PBL." *J. Atmos. Sci.,* **60,** 3043–3046.

Galperin, B., L. H. Kantha, S. Hassid, and A. Rosati, 1988: A quasi-equilibrium turbulent energy model for geophysical flows. *J. Atmos. Sci.,* **45,** 55–62.

——, S. Sukoriannsky, and P. S. Anderson, 2007: On the critical Richardson number in stably stratified turbulence. *Atmos. Sci. Lett.,* **8,** 65–69.

Kantha, L. H., 2003a: On an improved model for the turbulent PBL. *J. Atmos. Sci.,* **60,** 2239–2246.

——, 2003b: Reply. *J. Atmos. Sci.,* **60,** 3047–3049.

——, 2006: Comments on "Second-order turbulence closure models for geophysical boundary layers: A review of recent work." *Cont. Shelf Res.,* **26,** 819–822.

——, and C. A. Clayson, 1994: An improved mixed layer model for geophysical applications. *J. Geophys. Res.,* **99** (C2), 25 235–25 266.

Mellor, G. L., and T. Yamada, 1982: Development of a turbulence closure model for geophysical fluid problems. *Rev. Geophys.,* **20,** 851–875.

Sukoriansky, S., B. Galperin, and I. Staroselsky, 2005: A quasi-normal scale elimination model of turbulent flows with stable stratification. *Phys. Fluids,* **17,** 1–28.

Umlauf, L., and H. Burchard, 2003: A generic length-scale equation for geophysical turbulence models. *J. Mar. Res.,* **61,** 235–265.

——, and ——, 2005: Second-order turbulence closure models for geophysical boundary layers. A review of recent work. *Cont. Shelf Res.,* **25,** 795–827.

Zilitinkevich, S. S., T. Elperin, N. Kleeorin, and I. Rogachevskii, 2007: Energy- and flux-budget (EFB) turbulence closure model for stably stratified flows. Part I: Steady-state, homogeneous regimes. *Bound.-Layer Meteor.,* **125,** 167–191.

——, ——, ——, ——, I. Esau, T. Mauritsen, and M. W. Miles, 2008: Turbulence energetics in stably stratified geophysical flows: Strong and weak mixing regimes. *Quart. J. Roy. Meteor. Soc.,* **134,** 793–799.